

Monopole condensation and colour confinement

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The evidence is reviewed for the mechanism of colour confinement in QCD by dual superconductivity of the ground state, i.e. by condensation of monopoles.

§1. Introduction

Most of the existing tests of Quantum Chromodynamics (QCD) as theory of strong interactions come from short distance phenomena (Deep inelastic scattering, $e^+ e^- \rightarrow$ hadrons, Jet physics, ...). Perturbation theory is supposed to work in that regime because of asymptotic freedom.

On the other hand it is known that the renormalized perturbative expansion does not converge, not even as an asymptotic series, due to bad infrared behaviour¹⁾. Fock vacuum is unstable: quarks and gluons, which are the elementary excitations of perturbation theory, never appear as asymptotic states. This phenomenon is known as colour confinement.

A convincing experimental evidence for colour confinement is the upper limit on the cosmic abundance of relic quarks, n_q :

$$\frac{n_q}{n_p} < 10^{-27} \quad (1.1)$$

n_p is the abundance of nucleons.

Eq.(1.1) correspond to Millikan like analysis of $\sim 10^2$ gr of matter. For non confined quarks the standard cosmological model predicts

$$\frac{n_q}{n_p} < 10^{-12} \quad (1.2)$$

A non perturbative formulation of the theory is needed, as well as a theoretical understanding of why perturbation theory works at all at short distances.

Lattice formulation provides that formulation.

The Feynman path integral defining the theory is regularized by discretizing space time, and computed numerically by Montecarlo techniques. Of course numerical computations have not the logical transparency of mathematical derivations, but they can help understanding anyhow. Lattice can be used of course to compute observable quantities (masses, matrix elements...) from first principles (Lattice for phenomenology), but also as a tool to explore mechanisms and structures of the

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theory by means of “Gedanken experiments” on the configurations produced by numerical simulations (Lattice for theory).

We will be concerned with an investigation of the second type (Lattice for theory), and test on the lattice the idea²⁾⁻⁴⁾ that confinement of colour is produced by *Dual superconductivity of type II of the QCD vacuum*.

The basic idea is that the chromoelectric field produced by a $Q\bar{Q}$ pair is channeled by dual Meissner effect into Abrikosov flux tubes in the same way as magnetic field is confined in usual superconductors of type II.

The energy is proportional to distance

$$E = \sigma R \quad (1.3)$$

and this means confinement. σ is the string tension.

The world “dual” here means that the role of electric and magnetic quantities is interchanged with respect to ordinary superconductors.

The idea is theoretically appealing in many respects.

- 1) Superconductivity is a symmetry⁵⁾: deconfining transition is a change of symmetry. An order (disorder) parameter can be defined and used to explore superconductivity. Dual superconductivity means a condensation of monopoles in the ground state. Vacuum has no definite magnetic charge, but is a superposition of states with different values of it^{6), 7)}.

A disorder parameter for dual superconductivity will then be the vacuum expectation value (vev) of any operator carrying magnetic charge.

- 2) The very existence of monopoles implies that the theory is compact, because of the Dirac quantization condition. Topology plays then an important role, since Dirac strings make the connection of the gauge field non trivial. A formulation in terms of parallel transport, like Wilson’s lattice theory, is then superior with respect to the perturbative formulation in terms of local fields, which misses topology.
- 3) Monopole charges are always $U(1)$ ^{8), 9)}: in $SU(N)$ gauge theory there are $N - 1$ $U(1)$ magnetic charges. Monopole species are associated to any field Φ in the adjoint representation

$$\Phi(x) = \sum_a \Phi^a \lambda^a \quad (1.4)$$

λ^a are the generators of the gauge group in the fundamental representation.

In field configurations monopoles are located at the sites where two eigenvalues of the $N \times N$ matrix Φ , eq.(1.4), coincide. This is of course a gauge invariant statement. The $N - 1$ $U(1)$ fields associated with magnetic charges are exposed by a gauge transformation $U(x)$ which diagonalizes $\Phi(x)$. Such a transformation is called an abelian projection.

There are a functional infinity of choices for Φ , and to each of them monopole species are associated. What species do condense in the vacuum to produce superconductivity is a dynamical question, and is actually what has to be investigated to understand the mechanism of confinement. A possibility is that all species are equally important for confinement and condense in the vacuum¹⁰⁾.

One tool to investigate this issue is to define a disorder parameter for different species and to measure it in connection with the deconfining transition. As we shall see below this connection can be made unambiguous by a measurement of the critical indices or effective critical indices of the phase transition, extracted from the behaviour of the disorder parameter. A somewhat different attitude is to look at the abelian dominance¹¹⁾. For a given abelian projection physical quantities as the string tension are measured in the full theory and in the $U(1)$ theory resulting from the abelian projection. If the latter determination is a good approximation to the exact quantity, people say that there is abelian dominance. This happens to 80% approximation in the so called maximal abelian projection. In addition one can separate the abelian field into a part due to monopoles plus a residual part with no topology¹²⁾. If the monopole part dominates people say that there is monopole dominance. Again this happens in the maximal abelian projection. Monopole dominance is then considered as a strong indication that confinement is due to monopoles.

The two approaches are in our opinion both important to disentangle the structure of the theory.

Our strategy is the following

- a) We define a disorder parameter for $U(1)$ dual superconductivity, and test its construction in compact $U(1)$ gauge theory. The construction is also tested with other well known systems, like the XY 3d model, which describes the transition to superfluid He_4 .

In both cases we are able to detect the change of symmetry of the ground state, and to determine the critical indices.

- b) We then define a similar disorder parameter for abelian projected monopoles of non abelian gauge theories, and explore by it the occurrence of dual superconductivity in connection with confinement.

A systematic analysis is in progress. Our preliminary results confirm that monopoles defined by different abelian projection condense in the confined phase, and the superconductivity disappears in the quark gluon phase, supporting the view of ref.¹⁰⁾.

We will start these lectures by a brief introduction to basic superconductivity in sect.2.

In sect.3 we shall recall the main properties of monopoles and the concept of duality.

We will then describe the disorder parameter for condensation of monopoles in $U(1)$ gauge theory and for condensation of vortices in 3d XY model (sect.'s 4,5).

The abelian projection and the physical meaning of monopoles will be discussed in sect.6. The results for $SU(2)$, $SU(3)$ and the other evidences from lattice for dual superconductivity mechanism will be reviewed.

Sect.7 will summarize the state of the art and present the open problems.

§2. Superconductivity as a symmetry.

A relativistic version of the Ginzburg-Landau free energy, which is the statistical analog of effective action, is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + (D_\mu\Phi)^\dagger(D_\mu\Phi) - V(\Phi) \quad (2.1)$$

where Φ is a complex (charged) scalar field describing Cooper pairs of charge $q = 2e$.

$$D_\mu\Phi = (\partial_\mu - iqA_\mu)\Phi \quad (2.2)$$

is the covariant derivative,

$$V(\Phi) = \frac{\lambda}{4}(\Phi^\dagger\Phi - \mu^2) \quad (2.3)$$

the potential, with μ and λ functions of the temperature T , $\mu = \mu(T)$, $\lambda = \lambda(T)$.

Minimizing \mathcal{L} defines the ground state. If $\mu^2 > 0$ the minimum corresponds to some $\langle\Phi\rangle \neq 0$, or to the Higgs phase. At T where $\mu = 0$ a transition to normal phase ($\mu^2 < 0$, $\langle\Phi\rangle = 0$) takes place. $\langle\Phi\rangle$ is the order parameter of superconductivity.

Putting $\Phi = \rho e^{i\theta}$, $\rho > 0$, under gauge transformations of angle $\alpha(x)$

$$\rho(x) \rightarrow \rho(x) \quad \theta(x) \rightarrow \theta(x) + \alpha(x) \quad A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu\alpha$$

Therefore for the covariant derivative

$$D_\mu\Phi = e^{i\theta} [\partial_\mu - iq(A_\mu - \partial_\mu\theta)] \rho$$

$\tilde{A}_\mu = (A_\mu - \partial_\mu\theta)$ is gauge invariant and $F_{\mu\nu} = \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu$, since $\partial_\mu\theta$ does not contribute.

\mathcal{L} can be rewritten as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{m^2}{2}\tilde{A}_\mu\tilde{A}_\mu + \tilde{\mathcal{L}}[\rho] \quad (2.4)$$

and the equations of motion for the electromagnetic field read

$$\partial_\mu F_{\mu\nu} + m^2\tilde{A}_\nu = 0 \quad (2.5)$$

with $m^2 = 2q^2\langle\Phi\rangle^2$.

In a stationary state with no charges $A_0 = 0$, $\partial_0\vec{A} = 0$ and equation (2.5) gives ($\vec{H} = \vec{\nabla} \wedge \vec{A}$)

$$\vec{\nabla} \wedge \vec{H} + m^2\vec{\tilde{A}} = 0 \quad (2.6)$$

$$\nabla^2\vec{H} + m^2\vec{H} = 0 \quad (2.7)$$

Eq.(2.6) means that a permanent current (London current) $\vec{j} = m^2\vec{\tilde{A}}$ exists, and since $\vec{E} = 0$ and $\vec{E} = \rho\vec{j}$, $\rho = 0$. If m^2 , or $\langle\Phi\rangle$ is different from zero, there is superconductivity ($\rho = 0$)

Eq.(2.7) is nothing but Meissner effect: the penetration depth of the field \vec{H} is $\lambda = 1/m$ and is again finite if $\langle\Phi\rangle \neq 0$.

A side consequence of Meissner effect is flux quantization: outside a flux tube, at distances larger than λ , $\vec{A} = 0$. The integral around a circle C centered on the section of the tube of \vec{A} is zero

$$0 = \oint \vec{A} d\vec{x} = \oint (\vec{A} - \nabla\theta) d\vec{x}$$

or, since the flux $F(H) = \oint \vec{A} d\vec{x}$

$$F(H) = \frac{2\pi n}{q} \quad (2.8)$$

The key parameter is the order parameter $\langle\Phi\rangle$ which signals the Higgs phenomenon.

$\langle\Phi\rangle \neq 0$ means condensation of charge. Indeed, if the ground state has a definite charge, the expectation value on it of any charged operator C is zero: $\langle 0|C|0\rangle = 0$, since $C|0\rangle$ belongs to a different eigenvalue of the charge than $|0\rangle$.

Superconducting vacuum is indeed known to be a coherent superposition of states with different numbers of Cooper pairs^{(6), (7)}.

There are two characteristic lengths in the system: the correlation length of the Φ field, or the inverse Higgs mass $\Lambda = 1/M$, and the penetration depth of the photon, λ . If $\lambda > \sqrt{2}\Lambda$ the superconductor is called type II, and the formation of Abrikosov flux tubes is favoured in the process of penetrating the material with a magnetic field⁽¹³⁾. If the opposite inequality holds, $\lambda < \sqrt{2}\Lambda$, when the magnetic field is increased there is an abrupt penetration of it at some value and superconductivity is destroyed.

In principle many independent charged fields could condense in the vacuum. In that case

$$\frac{m^2}{2} = \sum q_i^2 \langle\Phi_i\rangle^2 \quad (2.9)$$

§3. Monopoles and their topology.

3.1. $U(1)$ monopoles.

Maxwell's equations in the presence of both electric and magnetic currents, j^μ , j_M^μ , are

$$\partial_\mu F^{\mu\nu} = j^\nu \quad \partial_\mu F^{*\mu\nu} = j_M^\nu \quad (3.1)$$

$F_{\mu\nu}$ is the familiar field strength tensor, $F_{\mu\nu}^*$ its dual

$$F_{\mu\nu}^* = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{*\rho\sigma}$$

If both j^ν and j_M^ν are zero the transformation

$$F_{\mu\nu} \rightarrow \cos\theta F_{\mu\nu} + \sin\theta F_{\mu\nu}^* \quad (3.2)$$

$$F_{\mu\nu}^* \rightarrow \cos\theta F_{\mu\nu}^* - \sin\theta F_{\mu\nu} \quad (3.3)$$

is a symmetry of the system for any value of θ . In particular for $\theta = \pi$ the transformation becomes

$$\vec{E} \rightarrow \vec{H} \quad \vec{H} \rightarrow -\vec{E}$$

which is known as duality transformation.

In nature, within the present experimental limits, $j_M^\mu = 0$, since no isolated magnetic charge has been found. Therefore

$$\partial_\mu F^{*\mu\nu} = 0 \quad (3.4)$$

the general solution of this equation is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (3.5)$$

which makes eq.(3.4) identically satisfied (Bianchi identity).

Of course if $j_M^\mu \neq 0$ Bianchi identity is violated, and the vector potential A_μ cannot be defined. The way out of this difficulty, was proposed by Dirac¹⁴⁾. A monopole can be seen as the end point of a magnetic flux tube, an infinitely thin solenoid (Dirac string), thus preserving Bianchi identities. The solenoid will be physically invisible if for any particle with charge e the parallel transport around it is trivial

$$\exp(ie \oint \vec{A} d\vec{x}) = 1 \quad \text{or} \quad \Phi \cdot e = 2\pi n \quad (3.6)$$

The magnetic flux on the other hand is related to the visible magnetic charge M of the monopole as $\Phi = M$, and this requires

$$eM = 2\pi n \quad (3.7)$$

with n integer, for any particle. Any charge is then multiple of the same elementary charge

$$e = \frac{2\pi}{M} \quad (3.8)$$

Theory is compact. A formulation in terms of parallel transport is then superior in that it naturally describes the non trivial connection of space time produced by the presence of Dirac strings.

One could formulate the theory in a dual form, by introducing a dual vector potential, and then $\partial_\mu F_{\mu\nu} = 0$ play the role of Bianchi identities. In this formulation charges would acquire a Dirac string of electric flux, and monopoles would be pointlike.

Due to Dirac quantization condition, the weak coupling regime with respect to charge corresponds to large values of M , or to strong coupling in the dual language and viceversa. This is typical of systems with topological excitations.

The prototype model is the 2d Ising model. The field variable is $\sigma(i) = \pm 1$ and the action

$$\beta J \sum_{i,\hat{\mu}} \sigma(i) \sigma(i + \hat{\mu})$$

Looking at it as a 1+1 dimensional field theory configurations like the one in fig.1 are kinks and have a non trivial topology. One can define a dual lattice, by associating a point i' of it to each link of the original lattice and a field $\sigma^*(i') = \pm 1$, assuming the value +1 if the sites at the ends of the link have opposite sign, and -1 if they

have the same sign. For a kink like the one in figure σ^* is +1 at the position of the kink, -1 everywhere else.

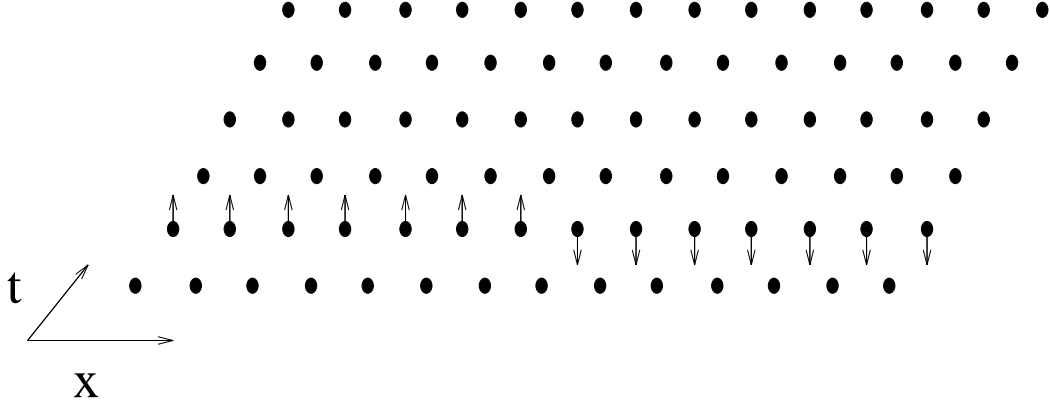


Fig.1 A kink in 2d Ising model.

Duality in this case¹⁵⁾ means that the partition function of the dual system has the same form of the original one,

$$K[\sigma, \beta] = K[\sigma^*, \beta^*] \quad (3.9)$$

with the change $\beta \rightarrow \beta^* \sim 1/\beta$.

The two systems are identical, but the low temperature (weak coupling) regime of one of them corresponds to the high temperature (strong coupling) of its dual. In the ordered phase $\langle \sigma \rangle \neq 0$, $\langle \sigma^* \rangle = 0$, and in the disordered phase $\langle \sigma^* \rangle \neq 0$, $\langle \sigma \rangle = 0$. $\langle \sigma^* \rangle$ is called a disorder parameter.

3.2. Monopoles in non abelian gauge theories.

Monopoles can exist as solitons, or static solutions with finite energy, in non abelian gauge theories. They were discovered^{16), 17)} in the Georgi-Glashow model, a gauge theory with gauge group $SO(3)$ coupled to a triplet of scalar fields $\vec{\phi}$: the simplest generalization of the $U(1)$ Higgs model. The lagrangean is

$$\mathcal{L} = -\frac{1}{4}\vec{G}_{\mu\nu}\vec{G}_{\mu\nu} + (D_\mu\vec{\phi})^\dagger(D_\mu\vec{\phi}) - V(\vec{\phi}) \quad (3.10)$$

with D_μ the covariant derivative

$$D_\mu\vec{\phi} = (\partial_\mu - g\vec{A}_\mu \wedge)\vec{\phi}$$

and $V(\vec{\phi})$ the potential

$$V(\vec{\phi}) = \frac{\lambda}{4}(\vec{\phi}^2 - \mu^2)^2$$

In the spontaneously broken phase, $\mu^2 > 0$, $\phi_0 = |\vec{\phi}_0| = \mu$. The ansatz

$$\partial_0\vec{\phi} = 0 \quad \vec{\phi}(\vec{r}) = f(r)\phi_0 \hat{r} \quad \hat{r} = \frac{\vec{r}}{r} \quad (3.11)$$

$$\vec{A}_0(\vec{r}) = 0 \quad A_i^a(\vec{r}) = h(r)\varepsilon_{iab}\frac{r_b}{gr^2} \quad (3.12)$$

brings to a solution with finite energy, with

$$h(r) \xrightarrow{r \rightarrow \infty} 1 \quad f(r) \xrightarrow{r \rightarrow \infty} 1$$

The configuration is called an hedgehog, due to the form (3.11) of the Higgs field.

That this configuration is a monopole can be seen by the following arguments.

- a) At large distances, where $h(r) = f(r) = 1$ a gauge transformation which brings $\hat{\phi}$ on some direction, say the 3 axis, (unitary gauge) transforms the gauge field to an abelian field, parallel to $\vec{\phi}$, with a Dirac string in space along the 3 axis.
- b) A gauge invariant “electromagnetic field” can be defined¹⁶⁾

$$\mathcal{F}_{\mu\nu} = \hat{\phi} \cdot \vec{G}_{\mu\nu} - \frac{1}{g} \hat{\phi} \left(D_\mu \hat{\phi} \wedge D_\nu \hat{\phi} \right) \quad (3.13)$$

Here $\hat{\phi} \equiv \vec{\phi}/|\vec{\phi}|$.

$\mathcal{F}_{\mu\nu}$, computed on the monopole configuration, is

$$(\vec{E})_i = \mathcal{F}_{0i} = 0 \quad \vec{H} = \frac{1}{g} \frac{\hat{r}}{r^2} + \text{Dirac string}$$

\vec{H} is a colour singlet, and such are the magnetic charges.

The gauge transformation bringing to the unitary gauge is called abelian projection. For the monopole configuration it has a singularity at $\vec{r} = 0$, where $\hat{\phi}$ is not defined.

An alternative way to look at the problem is to use the Body Fixed Frame (BFF)¹⁸⁾. Usually the same reference frame for colour is used in all points of space time, $\vec{\xi}_0^i$, with $\vec{\xi}_0^i \vec{\xi}_0^j = \delta^{ij}$, $\vec{\xi}_0^i \wedge \vec{\xi}_0^j = \vec{\xi}_0^k$. Instead three orthonormal unit vectors $\vec{\xi}_i(x)$ can be defined (with $\vec{\xi}_i^i \vec{\xi}_i^j = \delta^{ij}$, $\vec{\xi}_i^i \wedge \vec{\xi}_i^j = \vec{\xi}_i^k$) with $\vec{\xi}^3(x) = \hat{\phi}(x)$. The choice of $\vec{\xi}_1(x)$, $\vec{\xi}_2(x)$ is arbitrary by an angle. The two frames are related by a transformation of $SO(3)$, $R(x)$

$$\vec{\xi}_i(x) = R(x) \vec{\xi}_i^0 \quad (3.14)$$

Since $(\vec{\xi}_i)^2 = 1$, $\partial_\mu \vec{\xi}_i$ is orthogonal to $\vec{\xi}_i$ and

$$\partial_\mu \vec{\xi}_i = \vec{\omega}_\mu \wedge \vec{\xi}_i \quad (3.15)$$

or

$$(\partial_\mu - \vec{\omega}_\mu \wedge) \vec{\xi}_i \equiv D_\mu \vec{\xi}_i = 0 \quad (3.16)$$

Indeed the body fixed frame changes with x by a parallel transport.

Eq.(3.16) also implies

$$[D_\mu, D_\nu] \vec{\xi}_i = 0 \quad (3.17)$$

From the completeness of $\vec{\xi}_i$, $[D_\mu, D_\nu] = 0$. This means

$$\vec{G}_{\mu\nu}(\omega) = \partial_\mu \vec{\omega}_\nu - \partial_\nu \vec{\omega}_\mu + \vec{\omega}_\mu \wedge \vec{\omega}_\nu = 0 \quad (3.18)$$

$\vec{\omega}_\mu$ is a pure gauge, at least in the regions of space where $R(x)$ is not singular.

The solution of Eq.(3.16) is

$$\hat{\Phi}(x) \equiv \vec{\xi}_3(x) = P \exp \left[i \int_{\mathcal{C}}^x \vec{\omega}_\mu \cdot \vec{T} dx^\mu \right] \vec{\xi}_0^3 \quad (3.19)$$

which is independent of the choice of the path \mathcal{C} if $\vec{G}_{\mu\nu} = 0$. P means path ordering and \vec{T} are the generators of $SO(3)$ group, $(T^j)_{ik} = -i\varepsilon_{ijk}$.

Expressing $\vec{\xi}_i(x)$ in terms of the polar angles θ, ψ with respect to $\vec{\xi}_i^0$, with polar axis along $\vec{\xi}_3^0$, one easily finds

$$\vec{\omega}_\mu = \begin{pmatrix} \sin \theta(x) \partial_\mu \psi(x) \\ -\partial_\mu \theta(x) \\ -\cos \theta(x) \partial_\mu \psi(x) \end{pmatrix} \quad (3.20)$$

At $\theta = 0, \pi$, ψ is not defined and a singular part of $\vec{\omega}_\mu$ develops

$$\vec{\omega}_\mu^{sing} = \begin{pmatrix} 0 \\ 0 \\ \pm \partial_\mu \psi^{sing}(x) \end{pmatrix} \quad (3.21)$$

and with it a singular field strength tensor

$$\vec{F}_{\mu\nu}(\omega) = \pm (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \psi^{sing} \vec{\xi}_3(x) \quad (3.22)$$

$\vec{F}_{\mu\nu}$ is abelian and parallel to $\hat{\phi} = \vec{\xi}_3$.

For the static monopoles $\vec{F}_{\mu\nu}(\omega)$ is a Dirac string along the 3 axis at all times: the end point of the string is the location of the monopole, i.e. the zero of $\vec{\phi}(x)$.

The monopole is not an artefact of the abelian projection, but a topological feature of the $\vec{\phi}$ field configuration.

The monopole configuration described above is also a soliton, and behaves like a particle.

However singularities in $\vec{\omega}_\mu$ can exist also in the unbroken phase of the theory, at the zeros of $\vec{\phi}$, and they are monopoles. Again they are topological properties of the $\vec{\phi}$ field configurations.

Under infinitesimal gauge transformations $\exp(i\vec{\lambda}(x)\vec{T})$

$$\begin{aligned} \vec{\omega}_\mu &\rightarrow \vec{\omega}_\mu + \vec{\lambda} \wedge \vec{\omega}_\mu + \partial_\mu \vec{\lambda} \\ \vec{A}_\mu &\rightarrow \vec{A}_\mu + \vec{\lambda} \wedge \vec{A}_\mu - \frac{1}{g} \partial_\mu \vec{\lambda} \end{aligned}$$

so that

$$\vec{\omega}_\mu + g\vec{A}_\mu = g\vec{Z}_\mu \quad (3.23)$$

is covariant.

In the abelian projected gauge $\vec{\omega}_\mu = 0$ and $\vec{A}_\mu = \vec{Z}_\mu$. The field strength tensor can be computed, obtaining

$$\vec{G}_{\mu\nu}(Z) = \vec{G}_{\mu\nu}(A) + \frac{1}{g} \vec{F}_{\mu\nu}(\omega) \quad (3.24)$$

The gauge transformation which operates the abelian projection is singular, and that produces the additional term in $\vec{G}_{\mu\nu}$. Also

$$D_\mu(A)\hat{\phi} = (\vec{\omega}_\mu + g\vec{A}_\mu) \wedge \vec{\varphi} = g\vec{Z}_\mu \wedge \hat{\phi} \quad (3.25)$$

and as a consequence

$$\frac{1}{g}\hat{\phi}(D_\mu\hat{\phi} \wedge D_\nu\hat{\phi}) = g(\vec{Z}_\mu \wedge \vec{Z}_\nu) \cdot \hat{\phi} \quad (3.26)$$

Thus the t'Hooft e.m. gauge invariant tensor

$$\mathcal{F}_{\mu\nu} = \vec{\phi} \cdot \vec{G}_{\mu\nu} - \frac{1}{g}\hat{\phi}(D_\mu\hat{\phi} \wedge D_\nu\hat{\phi})$$

reads in the abelian projected gauge

$$\mathcal{F}_{\mu\nu} = \partial_\mu Z_\nu^3 - \partial_\nu Z_\mu^3 \quad (3.27)$$

since the non abelian term in $\vec{\phi} \cdot \vec{G}_{\mu\nu}$ is canceled by the additional term by virtue of eq.(3.26).

The gauge invariant tensor which describes the $U(1)$ field coupled to monopole charge coincides with the abelian field of the residual $U(1)$ after abelian projection. Because of that sometimes instead of associating monopoles to the field $\vec{\phi}$, people associate them to the gauge which puts $\hat{\phi}$ along the 3 axis.

Since in QCD the possible fields $\hat{\phi}$ which could define monopoles are many, and to each of them a different gauge transformation is associated as abelian projection, the statement could be made that monopoles are gauge dependent objects or gauge artefacts. This is a misuse of language.

The physical problem is to identify what $\vec{\phi}(x)$ fields are relevant to confinement, in that monopoles associated to them condense and produce dual superconductivity. Monopole species depend on the choice of $\vec{\phi}$. However the monopole charges and electromagnetic field associated to each $\vec{\phi}$ are gauge invariant concepts.

§4. A disorder parameter for dual superconductivity: compact $U(1)$ gauge theory.

In any field theory in which non trivial topological objects T exist, like monopoles or vortices, with a conserved topological charge, a creation operator can be defined for them.

The original construction goes back to ref. ¹⁵⁾, and has been developed in different forms by many authors ²⁰⁾⁻²⁴⁾. The basic idea is translation, in the sense of the elementary formula

$$e^{ipa}|x\rangle = |x+a\rangle \quad (4.1)$$

If $\Phi(x)$ is the field describing the theory and $\Phi_T(x, y)$ is the field configuration produced by, say, a monopole sitting at y , then the operator

$$\mu(\vec{y}, t) = \exp\left(i \int d^3x \Pi_\Phi(\vec{x}, t) \Phi_T(x, y)\right) \quad (4.2)$$

with $\Pi_\Phi(\vec{x}, t)$ the conjugate momentum to $\Phi(\vec{x}, t)$, creates a monopole at the site \vec{y} and time t .

Indeed in the Schrödinger representation $|\Phi(\vec{x})\rangle$

$$\mu(\vec{y}, t)|\Phi(\vec{x})\rangle = |\Phi(\vec{x}) + \Phi_T(\vec{x}, \vec{y})\rangle$$

We will focus on compact $U(1)$ gauge theory in 4d on lattice^{25), 26)}, where monopoles exist and magnetic charge is conserved.

The building block of the theory is the parallel transport along links of the lattice, exiting from site n in direction μ

$$U_\mu(n) = \exp(ieaA_\mu(n)) \equiv \exp(i\theta_\mu(n)) \quad (4.3)$$

The parallel transport along the plaquette, the elementary square in the plane $\mu\nu$, is then

$$\begin{aligned} \Pi_{\mu\nu} &= \exp(i\theta_{\mu\nu}(n)) \\ \theta_{\mu\nu}(n) &= \Delta_\mu\theta_\nu - \Delta_\nu\theta_\mu \simeq a^2 ef_{\mu\nu} \end{aligned} \quad (4.4)$$

The generating functional of the theory or partition function is

$$Z(\beta) = \int \prod_{n,\mu} \left(\frac{d\theta_\mu(n)}{2\pi} \right) \exp(-S) \quad (4.5)$$

We will choose for S the Wilson action

$$S = \beta \sum_{n,\mu < \nu} (1 - \cos \theta_{\mu\nu}(n)) \quad (4.6)$$

As $\beta \rightarrow \infty$ small values of $\theta_{\mu\nu}$ are important and

$$S \underset{\beta \rightarrow \infty}{\simeq} \frac{1}{4} \beta \sum \theta_{\mu\nu}^2 = \beta a^4 \sum e^2 f_{\mu\nu}^2 \quad (4.7)$$

which is the action for free photons if the identification is made $\beta = 1/e^2$. Compactness of the theory, i.e. the fact that angles $\theta_\mu(n)$ only appear as arguments of periodic functions in the action, makes $Z(\beta)$ and correlations functions of compact field variables invariant under the change

$$\theta_\mu(n) \rightarrow \theta_\mu(n) + f_\mu(n)$$

with arbitrary $f_\mu(n)$. A special case are gauge transformations $f_\mu(n) = \Delta_\mu \Phi$.

A critical β_c exists in the model, $\beta_c \simeq 1.0116$ ^{27), 28)}.

For $\beta > \beta_c$ the theory describes free photons. For $\beta < \beta_c$ electric charge is confined. Wilson loops obey the area law, and dual Meissner effect is observed.

Monopoles exist in this theory. Indeed, since

$$-\pi \leq \theta_\mu(n) \leq \pi$$

from eq.(4.4)

$$-4\pi \leq \theta_{\mu\nu}(n) \leq 4\pi \quad (4.8)$$

Since in the plaquette integer multiples of 2π are not visible, because of compactness, one can redefine $\theta_{\mu\nu}$ as

$$\theta_{\mu\nu} = 2\pi n_{\mu\nu} + \bar{\theta}_{\mu\nu} \quad -\pi \leq \bar{\theta}_{\mu\nu} \leq \pi \quad (4.9)$$

$\bar{\theta}_{\mu\nu}$ is the visible flux.

Now

$$\theta_{\mu\nu}^* = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \theta_{\rho\sigma}$$

obeys the Bianchi identities $\Delta_\mu \theta_{\mu\nu}^* = 0$ as can be trivially checked. The visible field, however, is $\bar{\theta}_{\mu\nu}$, because of compactness and can violate Bianchi identities because units of 2π can be formed which becoomes invisible. A monopole current can be defined as

$$\rho_\mu^M = -\frac{1}{6} \varepsilon_{\mu\nu\rho\sigma} \Delta_\nu n_{\rho\sigma} \quad (4.10)$$

and

$$\Delta_\mu \bar{\theta}_{\mu\nu} = \rho_\mu^M \quad (4.11)$$

Monopoles are identified as Dirac strings. Since ρ_μ^M is identically conserved, (see eq.(4.10)) strings are closed.

In the pioneering work of ref.(²⁵) such monopoles where numerically detected. Their density being higher in the confined phase, $\beta < \beta_c$ and dropping to zero above β_c , the density of monopoles was called an order parameter for the transition. Although the observed correlation between density and phase is phenomenological significant, a genuine disorder parameter for the system should be related to the symmetry of the ground state and should then be the vev of a magnetically charged operator. A candidate disorder parameter is the vacuum expectation value of the creation operator of a monopole.

In the continuum the general rule eq.(4.2) gives for that operator

$$\mu(\vec{y}, t) = \exp \left[i \int d^3x \vec{E}(\vec{x}, t) \frac{1}{e} \vec{b}(\vec{x} - \vec{y}) \right] \quad (4.12)$$

$\vec{E}(\vec{x}, t)$ being the conjugate momentum to the field $\vec{A}(\vec{x}, t)$, and $1/e \vec{b}$ the classical configuration corresponding to a monopole sitting at \vec{y}

$$\vec{A}_{mon}(\vec{x} - \vec{y}) = \frac{1}{e} \vec{b}(\vec{x} - \vec{y}) \quad (4.13)$$

the factor $1/e$ coming from Dirac quantization condition is explicitly exposed. With some choice of the classical gauge, putting the string along the unit vector \vec{n}

$$\frac{1}{e} \vec{b}(\vec{r}) = \frac{2\pi m}{e} \frac{\vec{r} \wedge \vec{n}}{r(r - \vec{r} \cdot \vec{n})} \quad (4.14)$$

A change of the classical gauge is reabsorbed in the definition (4.12) if \vec{E} obeys Gauss's law.

On the lattice

$$\Pi_i = \frac{1}{e} \text{Im} \Pi^{0i} = \frac{1}{e} \sin \theta^{0i}$$

so that the naive transcription of eq.(4.12) would be²¹⁾

$$\mu(\vec{y}, n_0) = \exp \left[\beta \sum_{\vec{n}} b^i(\vec{n} - \vec{y}) \sin \theta^{i0}(\vec{n}, n_0) \right] \quad (4.15)$$

The factor β comes from the $1/e$ in the magnetic charge times $1/e$ in the normalization of the plaquette to the electric field.

The above definition can be adapted to compactness, i.e. to give a shift of the angle instead of its sinus, as follows¹⁹⁾

$$\mu(\vec{y}, n_0) = \exp \left[\tilde{S}(\theta^{i0}(n_0) + b^i(\vec{n} - \vec{y})) - \tilde{S}(\theta^{i0}(n_0)) \right] \quad (4.16)$$

\tilde{S} is the sum of the density of action on the time slice n_0

$$\tilde{S} = \sum_{\vec{n}, \mu, \nu} \mathcal{L}[\theta_{\mu\nu}(\vec{n}, n_0)] \quad (4.17)$$

Since in the limit $a \rightarrow 0$, $\mathcal{L} \rightarrow \frac{\beta}{2} \sum_{\mu < \nu} \theta_{\mu\nu}^2$

$$\mu \underset{a \rightarrow 0}{\simeq} \sum_{\vec{n}} \theta^{i0}(\vec{n}, n_0) b_i(\vec{n} - \vec{y}) \beta + \sum_{\vec{n}} b_i^2(\vec{n} - \vec{y}) \quad (4.18)$$

and μ coincides with the naive definition modulo a constant coming from the last term.

If more monopoles or antimonopoles are created at time n_0 , b^i should be replaced by the classical field describing their configuration.

Correlation functions of monopoles and or monopoles antimonopoles can then be constructed.

We will focus on the correlation¹⁹⁾

$$\mathcal{D}(x_0) = \langle \bar{\mu}(\vec{0}, x_0) \mu(\vec{0}, 0) \rangle \quad (4.19)$$

between a monopole sitting at $\vec{0}$ in space at time 0, and an antimonopole at $\vec{0}$ and time x_0 (propagator of monopole field).

At large x_0 we expect, by cluster property

$$\mathcal{D}(x_0) \simeq A \exp(-M|x_0|) + \langle \mu \rangle^2 \quad (4.20)$$

Translation and C invariance make

$$\langle \bar{\mu}(\vec{0}, x_0) \rangle = \langle \mu(\vec{0}, 0) \rangle \equiv \langle \mu \rangle \quad (4.21)$$

$\langle \mu \rangle$ is our disorder parameter: $\langle \mu \rangle \neq 0$ signals spontaneous breaking of magnetic $U(1)$ and hence dual superconductivity, as discussed in sect.2.

The other important quantity in eq.(4.20) is M , which is the lowest mass of excitations carrying monopole charge. The effective scalar field producing dual superconductivity has a mass larger or equal to M , and hence knowledge of M is an important information to determine the type of dual superconductivity.

Before going to numerical results we will clarify our definition of μ , by analyzing in detail $\mathcal{D}(x_0)$. According to that definition eq.(4.20)

$$\mathcal{D}(x_0) = \frac{1}{Z[S]} Z[S + \Delta S] \quad (4.22)$$

The factor $1/Z[S]$ comes from the averaging procedure, $Z[S + \Delta S]$ is nothing but the partition function with a modified action: using the definition eq.(4.16) the modification consists in replacing

$$\tilde{S}(\theta^{i0}(0)) \rightarrow \tilde{S}[\theta^{i0}(0) + b^i(\vec{y})] \quad (4.23)$$

$$\tilde{S}(\theta^{i0}(n_0)) \rightarrow \tilde{S}[\theta^{i0}(n_0) - b^i(\vec{y})] \quad (4.24)$$

Since

$$\theta^{i0}(\vec{n}, 0) = -\theta^i(\vec{n}, 1) + \theta^i(\vec{n}, 0) + \theta^0(\vec{n} + \hat{i}, 0) - \theta^0(\vec{n}, 0)$$

the change implied by eq.(4.23) can be reabsorbed by a change of integration variables

$$\theta^i(\vec{n}, 1) \rightarrow \theta^i(\vec{n}, 1) + b^i(\vec{n}) \quad (4.25)$$

which leaves Z unchanged, because of compactness.

The result of this change is that $\theta^{i0}(0)$ is restored to the form it has in $Z[S]$, but at $n_0 = 1$

$$\theta^{ij}(\vec{n}, 1) \rightarrow \theta^{ij}(\vec{n}, 1) + \Delta_i b_j - \Delta_j b_i$$

A monopole field is added at $n_0 = 1$, in a form which is independent of the gauge choice for b_i . $\theta^i(\vec{n}, 1)$ also appears in $\theta^{i0}(\vec{n}, 1)$ and, the change of variables gives

$$\theta^{i0}(\vec{n}, 1) \rightarrow \theta^{i0}(\vec{n}, 1) + b_i(\vec{n}) \quad (4.26)$$

which is the same as (4.23), at $n_0 = 1$. We can now repeat the procedure, and the result will be a monopole at time $n_0 = 2$ and again a change of the form (4.26) at time 2. The procedure ends at $n_0 - 1$ where the change (4.25) is reabsorbed by the antimonopole, eq.(4.24). Our construction really produces a monopole at site $\vec{0}$, propagating from 0 to x^0 .

A direct determination of μ from eq.(4.20) at large value of x_0 is shown in fig.2.

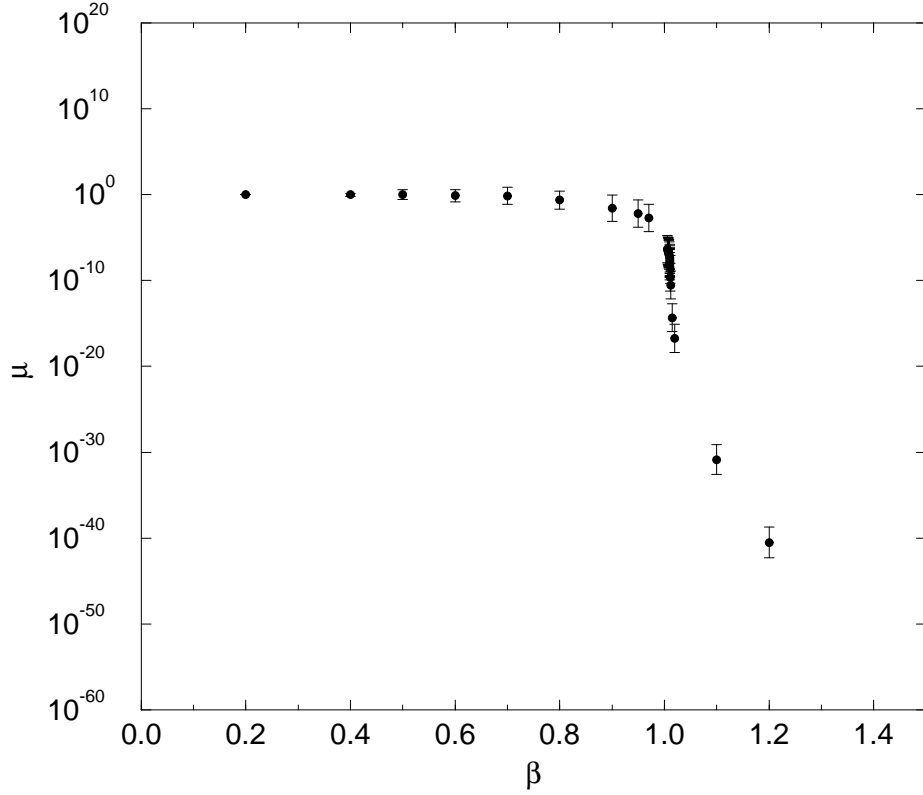


Fig.2 The disorder parameter vs β on a 10^4 lattice.

Instead of $\mathcal{D}(x_0)$ itself it provides numerically convenient to compute

$$\rho(x_0) = \frac{d}{d\beta} \ln \mathcal{D}(x_0) \quad (4.27)$$

or, by eq.(4.22)

$$\rho = \langle S \rangle_S - \langle S + \Delta S \rangle_{S+\Delta S}$$

The subscript of the brackets denotes the action used in performing average. Again as $|x_0| \rightarrow \infty$

$$\rho(x_0) \underset{|x_0| \rightarrow \infty}{\simeq} 2 \frac{d}{d\beta} \ln \langle \mu \rangle + C \exp(-M|x_0|) \quad (4.28)$$

The typical behaviour of $\rho(x_0)$ near the transition is shown in fig.3. The typical correlation length is of the order of the lattice spacing and M can be determined.

The behaviour of ρ_∞ is shown in fig.4. The sharp drop of $\langle \mu \rangle$ around β_c is reflected in a narrow negative peak in ρ .

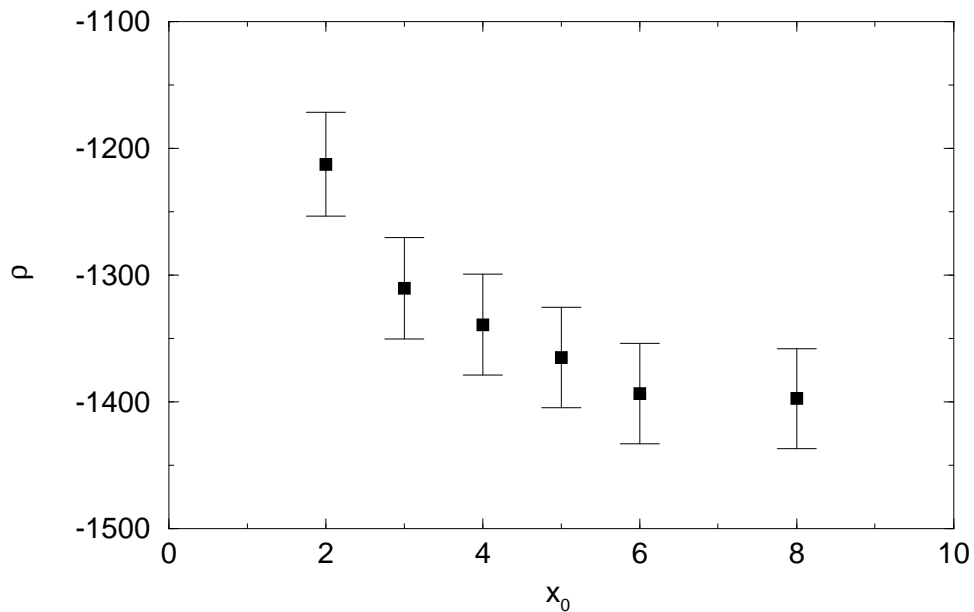


Fig.3 $\rho(x_0)$ vs x_0 at $\beta = 1.099$. The correlation length is of the order of lattice spacing.

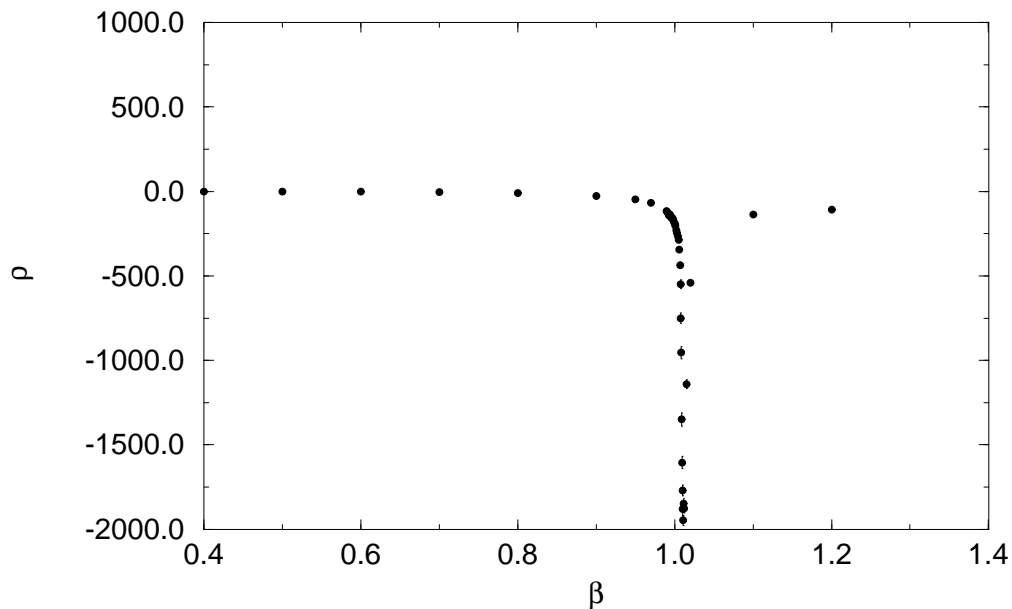


Fig.4 ρ as a function of β . The sharp negative peak indicates the phase transition.

For $\beta > \beta_c$ the system describes free photons, the Feynman integral is gaussian and ρ can be explicitly computed. Numerically one finds for a lattice $L^3 \times 2L$

$$\rho_\infty = -10.1 L + 9.542 \quad (4.29)$$

ρ_∞ tends to $-\infty$ as the volume goes large (thermodynamical limit). and

$$\mu = \exp \left(\int_0^\beta \rho(x) dx \right)$$

tends to zero. Above β_c magnetic $U(1)$ is not broken and $\langle \mu \rangle = 0$. Notice that this can only be true in the thermodynamical limit¹⁵⁾. In a finite volume $\langle \mu \rangle$ is an analytic function of β and cannot be identically zero on a line of the complex plane without being zero everywhere. Only as $L \rightarrow \infty$ Lee-Yang singularities develop and $\langle \mu \rangle$ can be zero.

For $\beta < \beta_c$ ρ_∞ tends to a finite value, compatible with zero as $L \rightarrow \infty$, and hence $\langle \mu \rangle \neq 0$ and the system is a dual superconductor.

The behaviour of $\langle \mu \rangle$ around β_c can be explored by a finite size scaling analysis. At $\beta \simeq \beta_c$ a weak first order or second order phase transition takes place. The order is controversial^{27), 28)}. In any case the correlation length ξ goes large in a range of values around β_c , and an effective critical index ν can be defined

$$\xi \underset{\beta \rightarrow \beta_c}{\simeq} (\beta_c - \beta)^{-\nu} \quad (4.30)$$

By dimensional analysis

$$\langle \mu \rangle = \mu \left(\frac{L}{\xi}, \frac{a}{\xi} \right)$$

as $\xi \rightarrow \text{large}$ the dependence on a/ξ can be neglected and

$$\langle \mu \rangle = \mu \left(\frac{L}{\xi}, 0 \right) = f(L^{1/\nu}(\beta_c - \beta)) \quad (4.31)$$

By use of eq.(4.30) the variable L/ξ has been traded with $L^{1/\nu}(\beta_c - \beta)$.

The following scaling law follows for $\rho = \frac{d}{d\beta} \ln \langle \mu \rangle$

$$\frac{\rho}{L^{1/\nu}} = -\frac{f'}{f} = \Phi(L^{1/\nu}(\beta_c - \beta)) \quad (4.32)$$

This scaling can be matched by appropriate values of ν and β_c . We find by best fit

$$\beta_c = 1.01160(5) \quad \nu = 0.29(2) \quad (4.33)$$

β_c agrees with determinations based on completely different methods²⁷⁾. The quality of scaling is shown in fig.5.

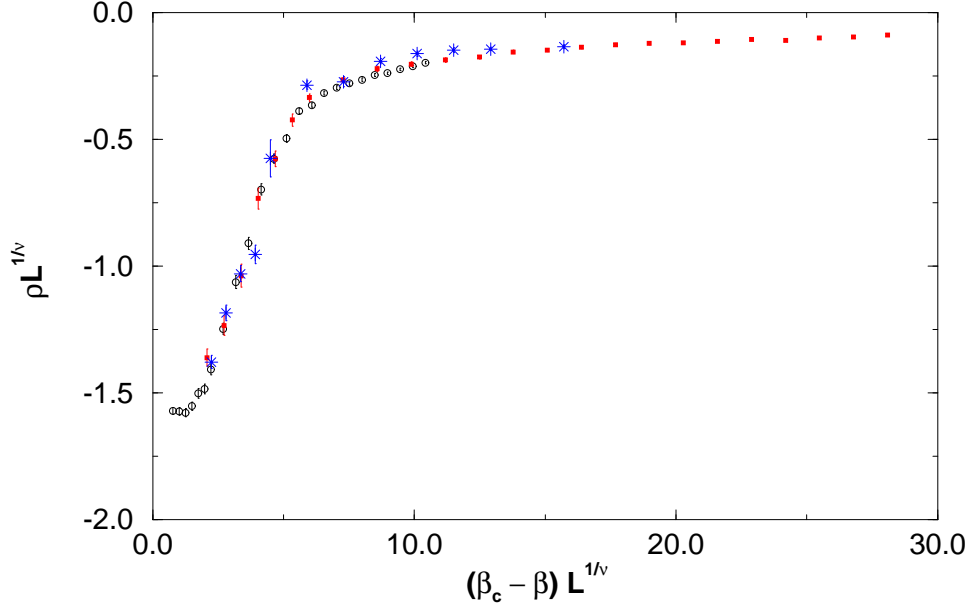


Fig.5 Finite size scaling of ρ , at the optimal values of ν , β_c .

We can also compute the critical index by which $\mu \rightarrow 0$ at β_c $\langle \mu \rangle \sim (\beta_c - \beta)^\delta$ finding

$$\delta = 1.1 \pm 0.2 \quad (4.34)$$

A first order transition would require $\nu = 1/d = 0.25$. Further investigation is on the way to test if the observed value of ν is a finite size effect or it is a true determination and the transition is 2nd order.

We have also measured the penetration depth of the field E , or the mass m of the photon. The result is shown in fig.6, where m is compared to M .

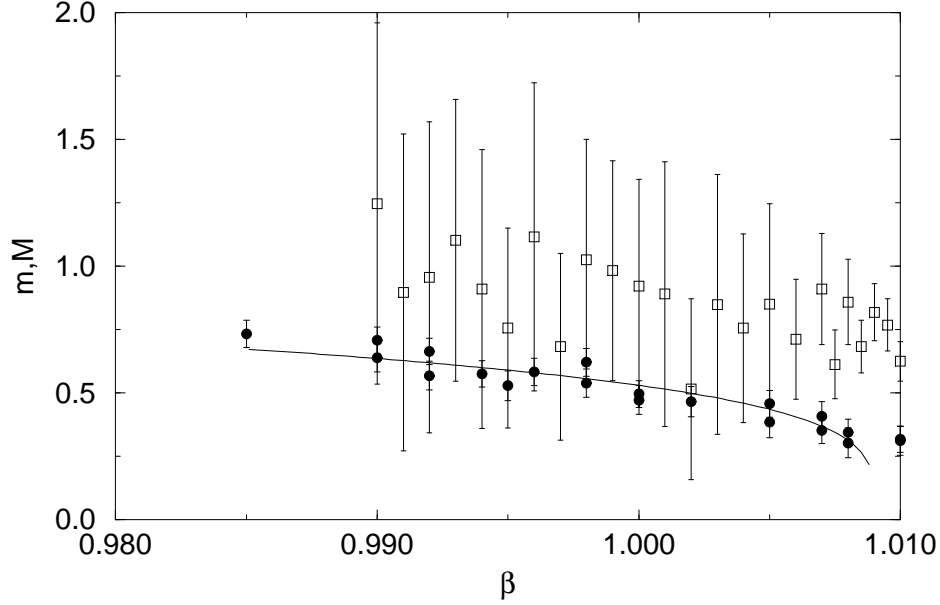
Further work is necessary to reduce the errors in M . There seems however to be sufficient information, at least in the region near β_c , to conclude that the ratio $M/m > \sqrt{2}$, i.e. that the superconductor is type II.

Similar results was found by direct observation of the London current in a flux tube, in ref.(²⁹).

If the transition is first order the lattice model does not define a field theory as $\beta \rightarrow \beta_c$ and the only fixed point is the trivial point $\beta = \infty$, which describes free photons. If this is the case changing the lagrangian from Wilson to an alternative form, say Villain, also changes the physics of the system, since there is no universality class.

For Villain action the duality transformation can be performed, and condensation of monopoles below the critical value has been proved(²³). In that case it can be shown as a theorem that our disorder parameter is equal to that of ref.(²³), although the construction is completely different. Both constructions are based on eq.(4.2). The proof of ref.(²³) has been extended to Wilson action in ref.(³⁰).

In conclusion we have a reliable tool to detect dual superconductivity.

Fig.6 m (circles) and M (squares) vs β .

§5. The XY model in 3d.

To further check our disorder parameter we have studied by the same technique used for $U(1)$ the XY model in 3d, which has a second order phase transition belonging to the same class of universality as the transition to superfluid He_4 .

The field variable is an angle $\theta(i)$ associated to each site. The action is

$$S = \beta \sum_i \sum_{\mu} [1 - \cos(\Delta_{\mu}\theta(i))] \quad (5.1)$$

and the partition function

$$Z = \int \prod_i \frac{d\theta(i)}{(2\pi)} \exp(-S) \quad (5.2)$$

The model is compact, so that any change $\theta(i) \rightarrow \theta(i) + f(i)$ with arbitrary $f(i)$ leaves correlation functions of compact observables invariant.

As $\beta \rightarrow \infty$ $S \simeq \frac{\beta}{2}(\Delta_{\mu}\theta)^2$ and the model describes free massless particles.

At $\beta_c \simeq 0.454$ a second order phase transition takes place, and at $\beta < \beta_c$ vortices condense. Like in $U(1)$ condensation has been demonstrated in the literature by a sharp change of density of vortices³²⁾. We shall show instead that for $\beta < \beta_c$ the $U(1)$ symmetry related to conservation of vortices is spontaneously broken, and we will construct a disorder parameter to detect the change of symmetry.

Let us define

$$A_{\mu} = \partial_{\mu}\theta \quad (5.3)$$

The invariance under $\theta \rightarrow \theta + f$ is a gauge invariance since on A_μ the transformation is

$$A_\mu \rightarrow A_\mu + \partial_\mu f$$

A_μ is a gradient, eq.(5.3), and hence a pure gauge

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0 \quad (5.4)$$

apart from singularities.

$\theta(x)$ can be written as a parallel transport

$$\theta(x) = \exp(ie \int_C^x A_\mu dx^\mu) \quad (5.5)$$

and if eq.(5.4) holds this definition is independent of the path C .

To investigate possible singularities consider the current

$$j_\mu = \varepsilon_{\mu\alpha\beta} \partial^\alpha A^\beta \quad (5.6)$$

which is dual to the field strength tensor. j_μ is identically conserved (Bianchi identity)

$$\partial^\mu j_\mu = 0$$

The corresponding charge is

$$Q = \int d^2x j_0(\vec{x}, t) \quad (5.7)$$

or

$$Q = \int d^2x (\vec{\nabla} \wedge \vec{A}) = \oint \vec{A} d\vec{x} = 2\pi n \quad (5.8)$$

The path on which the line integral of \vec{A} is computed is a circle at infinity: the value $2\pi n$ comes from the definition of A_μ , eq.(5.3).

In the absence of singularities $j_\mu \equiv 0$ and $Q = 0$. There exist, however, configurations with $Q \neq 0$, which are vortices. An example is

$$\tilde{\theta}_q(\vec{x} - \vec{y}) = q \operatorname{atan} \frac{(x - y)_2}{(x - y)_1} \quad (5.9)$$

For these configurations

$$A_0 = 0 \quad \vec{A} = \frac{q}{|\vec{x} - \vec{y}|} \vec{\nu}_\theta$$

where $\vec{\nu}_\theta$ is the unit vector tangent to the circle $|\vec{x} - \vec{y}| = r$. If the field \vec{A} is the field of velocities the configuration is a vortex with

$$Q = \oint \vec{A} d\vec{x} = 2\pi q \quad (5.10)$$

Topology is non trivial.

As a disorder parameter we will use the v.e.v. of the operator which creates a vortex, following the general rule eq.(4.2). Since the conjugate momentum to θ as given by the action is $\Pi = \beta \sin \partial_0 \theta$ the naive definition of μ would be

$$\mu(\vec{y}, t) = \exp \left[-\beta \sum_{\vec{n}} \sin(\Delta_0 \theta(\vec{n}, t)) \tilde{\theta}_q(\vec{n} - \vec{y}) \right] \quad (5.11)$$

The compact version is

$$\mu(\vec{y}, t) = \exp \left[\bar{S}(\Delta_0 \theta(t) - \tilde{\theta}_q) - \bar{S}(\Delta_0 \theta(t)) \right] \quad (5.12)$$

with \bar{S} the integral of the lagrangean on time slice t

$$\bar{S}(t) = \sum_{\vec{n}} S(\vec{n}, t)$$

Again, as for the $U(1)$, we compute the correlator

$$\mathcal{D}(x_0) = \langle \bar{\mu}(\vec{0}, x_0) \mu(\vec{0}, 0) \rangle \Big|_{|x_0| \rightarrow \infty} \simeq A \exp(-M|x_0|) + \langle \mu \rangle^2 \quad (5.13)$$

$\langle \mu \rangle$ is the disorder parameter which signals condensation of vortices, i.e. spontaneous breaking of the $U(1)$ symmetry. We have from the definition (5.12)

$$\mathcal{D}(x_0) = \frac{Z[S + \Delta S]}{Z[S]} \quad (5.14)$$

where $S + \Delta S$ is obtained from S by the replacements at time slices 0 and x_0 respectively

$$\Delta_0 \theta(\vec{n}, 0) \rightarrow \Delta_0 \theta(\vec{n}, 0) - \tilde{\theta}_q(\vec{n} - \vec{y}) \quad (5.15)$$

$$\Delta_0 \theta(\vec{n}, 0) \rightarrow \Delta_0 \theta(\vec{n}, x_0) + \tilde{\theta}_q(\vec{n} - \vec{y}) \quad (5.16)$$

The change (5.15) can be reabsorbed by a change of variables

$$\theta(\vec{n}, 1) \rightarrow \theta(\vec{n}, 1) + \tilde{\theta}_q \quad (5.17)$$

which leaves the measure unchanged. However it changes the space derivatives at time 1

$$\Delta_i \theta(\vec{n}, 1) \rightarrow \Delta_i \theta(\vec{n}, 1) + \Delta_i \tilde{\theta}_q \quad (5.18)$$

or

$$\vec{A}(\vec{n}, 1) \rightarrow \vec{A}(\vec{n}, 1) + \Delta_i \tilde{\theta}_q \quad (5.19)$$

thus adding a vortex to the configuration at $n_0 = 1$.

The other change in the action coming from the change of variables is

$$\Delta_0 \theta(\vec{n}, 1) \rightarrow \Delta_0 \theta(\vec{n}, 1) - \tilde{\theta}_q(\vec{n} - \vec{y}) \quad (5.20)$$

which is like eq.(5.15) with time 0 replaced by time 1. The change of variables can be repeated, producing a vortex at $n_0 = 2$ and a shift of $\Delta_0 \theta(\vec{n}, 2)$ and so on till $n_0 = x_0 - 1$ when the shift $-\tilde{\theta}_q$ of ΔS disappears with $+\tilde{\theta}_q$ of eq.(5.16).

Again it is numerically convenient to study

$$\rho(x_0) = \frac{d}{d\beta} \mathcal{D}(x_0) \Big|_{|x_0| \rightarrow \infty} \simeq \rho + C e^{-M|x_0|} \quad (5.21)$$

with

$$\rho = 2 \frac{d}{d\beta} \ln \langle \mu \rangle \quad \langle \mu \rangle = \exp \left(\frac{1}{2} \int_0^\beta \rho(\beta') d\beta' \right) \quad (5.22)$$

ρ as a function of β is shown in fig.7 for different lattice sizes.

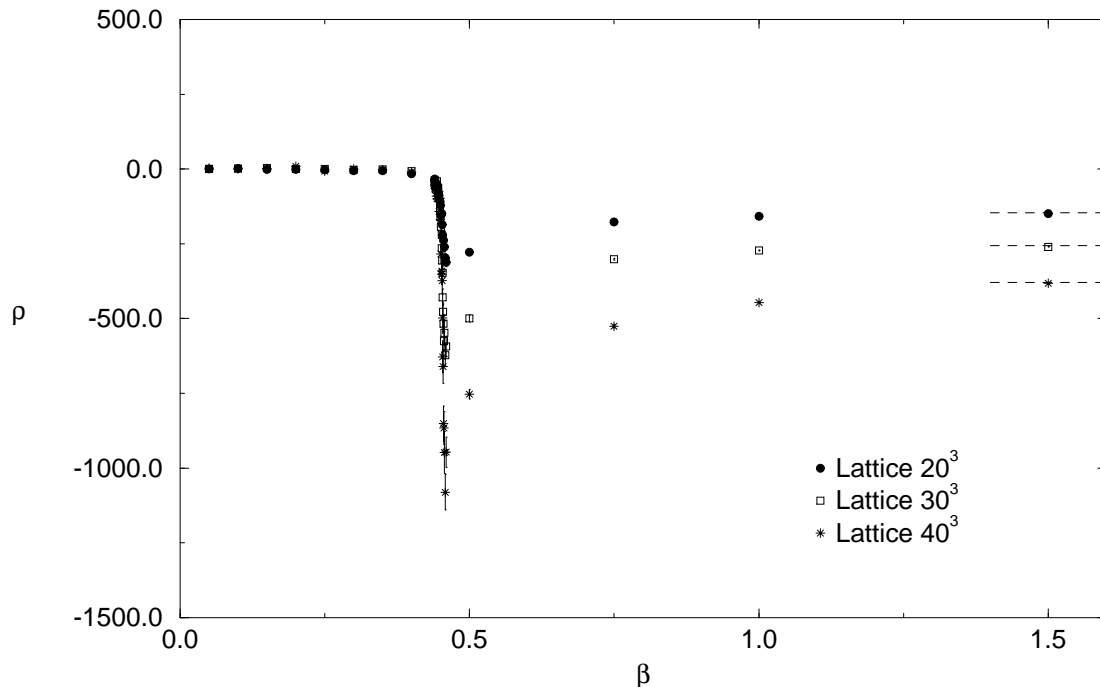


Fig.7 ρ vs β for different lattices. The peak signals the phase transition. The lines at high β are the comparison to perturbation theory, eq.(5.23).

A huge negative peak signals the phase transition. For $\beta > \beta_c$ the system describes free particles, the Feynman integral is gaussian and ρ can be computed giving

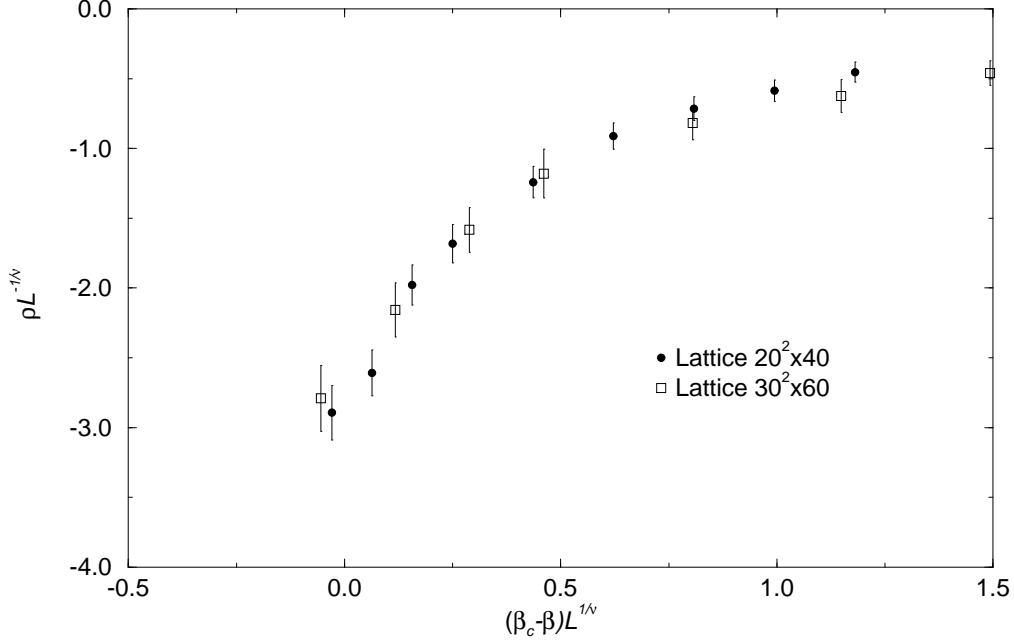
$$\rho = -11.33L + 72.7 \quad (5.23)$$

as $L \rightarrow \infty$, $\rho \rightarrow -\infty$, or $\langle \mu \rangle \rightarrow 0$, as expected in the ordered phase in the thermodynamical limit. The agreement with the prediction (5.23) at large β is shown in the figure: the value (5.23) is represented by the dotted lines.

For $\beta < \beta_c$ ρ tends, as $L \rightarrow \infty$, to a finite value compatible with zero, so that $\langle \mu \rangle \neq 0$: vortices do condense. A finite size scaling analysis around β_c gives the scaling law

$$\frac{\rho}{L^{1/\nu}} = f(L^{1/\nu}(\beta_c - \beta)) \quad (5.24)$$

ν and β_c can be adjusted to satisfy it. The quality of scaling is shown in fig.8

Fig.8 Finite size scaling for ρ in 3d XY model.

We get

$$\nu = 0.669 \pm 0.065[0.670(7)] \quad (5.25)$$

$$\beta_c = 0.4538 \pm 0.0003[0.4542(2)] \quad (5.26)$$

The values in square parantheses have been determined by completely different methods in the literature³³⁾, and are the critical indices of the transition to superfluid. The index δ of the order parameter

$$\langle \mu \rangle_{\beta \sim \beta_c} \simeq (\beta_c - \beta)^\delta \quad (5.27)$$

is also determined

$$\delta = 0.740 \pm 0.029 \quad (5.28)$$

Remarkable is the similarity of the shape of ρ in fig.7 and fig.4 in spite of the fact that the systems have nothing to do with each other.

Again we have checked that the construction works, and we will use it to study the deconfining phase transition as a possible transition from dual superconductor to normal.

§6. Monopole condensation vs confinement in QCD.

Contrary to the Georgi-Glashow model QCD has no fundamental Higgs field. However, as shown in sect.3 any local operator $\Phi(x)$ in the adjoint representation has monopoles associated with it, which are located at the zeros of $\Phi(x)$ for $SU(2)$. For gauge groups of higher rank, say $SU(3)$, $\Phi(x)$ is written as a matrix in the fundamental representation

$$\Phi(x) = \sum_a \Phi^a \lambda^a \quad (6.1)$$

with λ^a the generators in that representation, monopoles will be located at the sites where two eigenvalues of the matrix $\Phi(x)$ coincide, and will be identified by integer charges, in one to one correspondence with diagonal matrices of the algebra with integer or zero matrix elements^{8),9)}. For $SU(N)$ this means $N - 1$ $U(1)$ conserved monopole charges which can condense in the vacuum. For $SU(2)$ there is one charge. The treatment of higher groups does not add any conceptual point, but only formal complications. We will therefore use $SU(2)$ formulae to present our arguments.

The role of $\Phi(x)$, the operator which identifies monopoles, can be played a priori by infinitely many composite operators of the theory: actually by a functional infinity of them. Each of them defines monopoles, which can in principle condense and produce dual superconductivity. It is not understood a priori to our best knowledge, if all of these monopole species are really independent of each other and if many of them could condense at the same time in connection with confinement. It could also be that $U(1)$ superconductivity for many monopole species is a manifestation of a more clever mechanism, a really non abelian superconductivity. This would implement the guess by t'Hooft, that all monopole species, defined by any operator $\Phi(x)$ are physically equivalent¹⁰⁾.

What we can presently do is to investigate these issues on the lattice. Looking at the problem from the point of view of symmetry is the most direct way.

The choices which have been suggested in the literature for $\Phi(x)$ are¹⁰⁾

- 1) The Polyakov line, i.e. the parallel transport along the time axis to $+\infty$ and back from $-\infty$ via periodic boundary conditions.
- 2) Any component of the field strength.
- 3) $F_{\mu\nu}F_{\mu\nu}$, at least for $SU(3)$ when it has an octet part. For $SU(2)$ it is a singlet.
- 4) The operator which is implicitly defined by the maximization of the quantity, with respect to gauge variations^{10),34)}

$$\text{Tr} \left\{ \sigma_3 \Omega U_\mu(n) \Omega^\dagger \sigma_3 \Omega^\dagger U_\mu^\dagger(n) \Omega \right\} = \max \quad (6.2)$$

with $\Omega(n)$ a generic gauge transformation. This procedure defines an operator Φ in the adjoint representation which coincides with

$$\sum_\mu \left(U^\dagger(n) \sigma_3 U_\mu(n) + U_\mu(n - \hat{\mu}) \sigma_3 U_\mu^\dagger(n - \mu) \right) \quad (6.3)$$

in the maximal abelian gauge. The explicit form in a generic representation is not known.

To explore dual superconductivity we will construct a disorder parameter, as the v.e.v. $\langle \mu \rangle$ of an operator μ with $U(1)$ magnetic charge, as we have done for $U(1)$ and XY model.

Whatever the choice of $\vec{\Phi}$, in the gauge in which $\vec{\Phi} \cdot \vec{\sigma}$ is diagonal, i.e. after abelian projection, any link $U_\mu(n)$ can be written³⁴⁾

$$U_\mu(n) = e^{i\sigma_3\alpha_\mu(n)} e^{i\sigma_2\gamma_\mu(n)} e^{i\sigma_3\beta_\mu(n)} \left[e^{i\sigma_3\alpha_\mu(n)} e^{i\sigma_2\gamma_\mu(n)} e^{-i\sigma_3\alpha_\mu(n)} \right] e^{i\sigma_3(\alpha_\mu(n)+\beta_\mu(n))} \quad (6.4)$$

$e^{i\sigma_3\theta_\mu(n)} \equiv e^{i\sigma_3(\alpha_\mu(n)+\beta_\mu(n))}$ is the abelian link, which parallel transports the $U(1)$ field related to the monopole charges defined by $\vec{\Phi}$.

The creation operator of a monopole at time $t = 0$ will be defined by changing the kinetic term of the action at that time by adding the field of the monopole to $\theta^{i0}(n)$ ³⁵⁾.

The change on the $i - 0$ plaquette Π_{i0} will be

$$\Pi_{i0}(\vec{n}, 0) \rightarrow \Pi'_{i0}(\vec{n}, 0)_b$$

$$\begin{aligned} \Pi_{i0}(\vec{n}, 0) &= \text{Tr} \left[U_i(\vec{n}, 0) U_0(\vec{n} + \hat{i}, 0) U_i^\dagger(\vec{n}, 1) U_0^\dagger(\vec{n}, 0) \right] \\ \Pi'_{i0}(\vec{n}, 0)_b &= \text{Tr} \left[U'_i(\vec{n}, 0) U_0(\vec{n} + \hat{i}, 0) U_i^\dagger(\vec{n}, 1) U_0^\dagger(\vec{n}, 0) \right] \end{aligned}$$

and

$$U'_i(\vec{n}, 0) = e^{i\Lambda(n)\sigma_3} U_i(\vec{n}, 0) e^{ib_i^\perp \sigma_3} e^{-i\Lambda(n)\sigma_3} \quad (6.5)$$

We have operated the separation of the vector potential describing the monopole into transverse and longitudinal part

$$b_i(n) = b_i^\perp(n) + \partial_i A(n) \quad \partial_i b_i^\perp(n) = 0$$

The gauge part $e^{i\Lambda(n)\sigma_3}$ and $e^{i\Lambda(n+1)\sigma_3}$ can be reabsorbed by a rotation of the U_0 's which leaves the functional measure invariant, so that the definition is independent on the choice of the classical gauge for $b_i(n)$, and the net effect is to add $b_i^\perp(n)$ to the abelian phase of $U_i(\vec{n}, 0)$.

Consider now the correlator

$$\mathcal{D}(t) = \langle \bar{\mu}(\vec{x}, t) \mu(\vec{x}, 0) \rangle \quad (6.6)$$

As usual we can write

$$\mathcal{D}(t) = \frac{Z[S + \Delta S]}{Z[S]} \quad (6.7)$$

where $S + \Delta S$ is obtained by the substitution

$$\begin{aligned} \Pi^{0i}(\vec{n}, 0) &\rightarrow \Pi_b^{0i}(\vec{n}, 0) \\ \Pi^{0i}(\vec{n}, x_0) &\rightarrow \Pi_{-b}^{0i}(\vec{n}, x_0) \end{aligned}$$

The subscript b , $-b$ is to recall the sign of the monopole charge.

The effect of this procedure is to have a monopole created at $t = 0$, which propagates to time t , when it disappears. The construction to show this is identical to $U(1)$ if Φ is the Polyakov line. There after abelian projection the temporal links are diagonal

$$U_0(n) = e^{i\sigma_3\alpha_0(n)}$$

and the operator $e^{i\sigma_3 b_i^\perp}$ in the definition of U_i' commutes with them.

Then a change of variables from U_i to U_i' which leaves the measure invariant brings Π_{i0} back to the original form, but changes $\Pi_{ij}(\vec{n}, 1)$ by adding a monopole to the abelian field

$$\Delta_i\theta_j - \Delta_j\theta_i \rightarrow \Delta_i\theta_j - \Delta_j\theta_i + \Delta_i b_j - \Delta_j b_i$$

The other change is to $\Pi^{i0}(\vec{n}, 1) \rightarrow \Pi_b^{i0}(\vec{n}, 1)$. The procedure can then be repeated till at time t the change is reabsorbed by $-b$.

If the U_0 's are not diagonal in the abelian projected gauge, the construction is the same modulo additional parallel transports in the change of variables, which do not modify its content.

Again one can either measure \mathcal{D} itself or

$$\rho(t) = \frac{d}{d\beta} \ln \mathcal{D}(t) = \langle S \rangle_S - \langle S + \Delta S \rangle_{S+\Delta S} \quad (6.8)$$

One expects

$$\mathcal{D}(t) = \langle \mu \rangle^2 + A e^{-Mt} \quad (6.9)$$

$$\rho(t) \simeq \rho + C e^{-Mt} \quad (6.10)$$

$$\rho = 2 \frac{d}{d\beta} \ln \langle \mu \rangle \quad (6.11)$$

A similar analysis to the one performed for $U(1)$ brings to the determination of $\langle \mu \rangle$ in the thermodynamical limit, and of M . $\langle \mu \rangle \neq 0$ signals dual superconductivity. A measurement of the penetration depth of the field allows to establish the type of superconductor.

At finite temperature, i.e. keeping the time extension N_T of the lattice much smaller than the space extension N_S , the correlation of $\langle \mu \rangle$ to confinement can be studied. There a single monopole is used, and $\langle \mu \rangle$ is measured. A careful examination of the construction given above shows that, when computing $Z[S + \Delta S]$ of eq.(6.6) if periodic boundary conditions are used, the change of variable described above adds more and more monopoles when we go trough the boundary in time. The way to have one monopole is to use antiperiodic boundary conditions in time. This suggests that monopoles behave as fermions.

We are systematically exploring by our disorder parameter the deconfining transition in $SU(2)$ and $SU(3)$, by different choices for $\vec{\Phi}$, the field which defines monopoles. We are measuring penetration depths and critical indices. As for Φ we consider Polyakov line, field strength component and max abelian projection. Typical behaviour for ρ are shown in fig.9 and fig.10 for $SU(2)$ and $SU(3)$

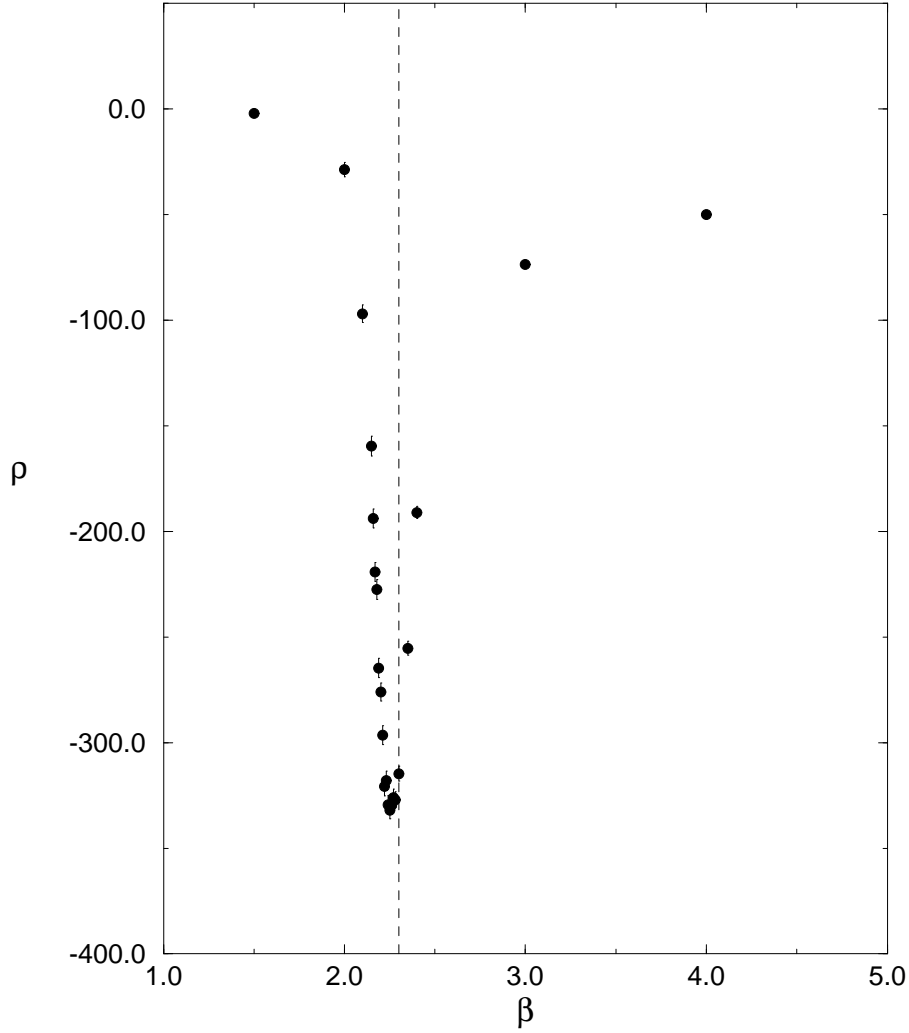
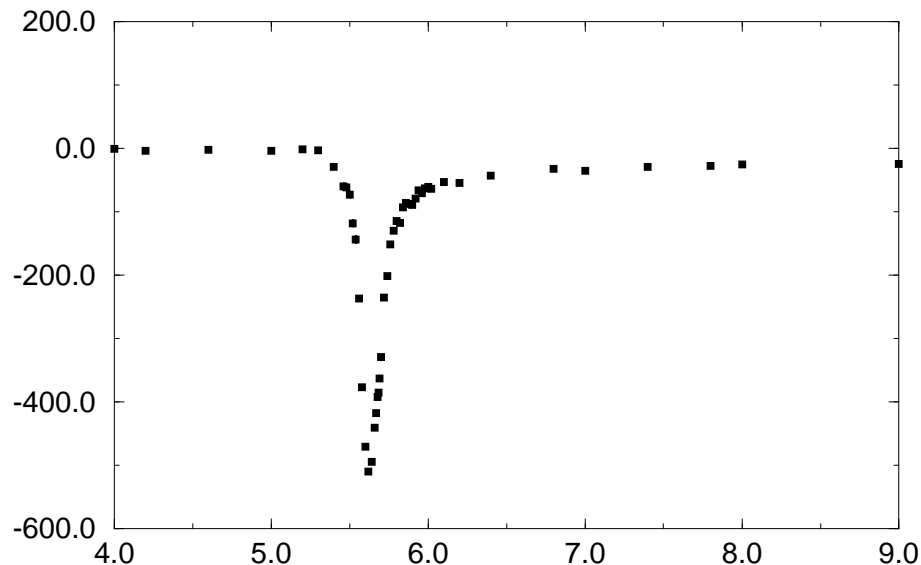


Fig.9 ρ vs β for $SU(3)$ gauge theory. The peak signals deconfining phase transition. Here monopoles are defined by the abelian projection on Polyakov line.

Preliminary evidence is that for all the species of monopoles considered, vacuum behaves as a dual superconductor, and undergoes a phase transition to normal at the deconfinement point. This supports the guess of t'Hooft about physical equivalence of different monopole species¹⁰⁾.

Fig.10 Same as Fig.9 for $SU(2)$.

§7. Concluding remarks.

Our strategy to answer the question if dual superconductivity is the mechanism of colour confinement, is to look at the symmetry of the vacuum. For that we have constructed a disorder parameter, which directly detects dual superconductivity.

The construction has been tested in known systems, like the $U(1)$ compact gauge theory and the XY model in 3 dimensions.

We have evidence that for many choices of the effective field $\Phi(x)$ defining monopole species, dual superconductivity is present in the confined phase, and disappears in the quark gluon phase.

Additional relevant information from lattice is that

- 1) Flux tubes exist in the space between propagating $Q\bar{Q}$ pair^{36), 37)}.
- 2) If one single species of monopoles were at work to produce superconductivity, then the electric field in the Abrikosov tubes should be the field of the $U(1)$ group to which monopole charges belong. An analysis of the colour content of the flux tubes shows instead that its direction in colour space is uncorrelated to the direction of $\vec{\Phi}$ ^{37), 38)}.

Moreover, whatever the abelian projection is, there exist one gluon in $SU(2)$, two of them in $SU(3)$, which have zero electric charge with respect to the residual $U(1)$'s, and therefore cannot be confined. The adjoint string tension is zero, and also this fact seems to contradict lattice observations³⁷⁾.

A different approach to the problem is to pay less attention to symmetry, and look at more quantitative facts, like abelian dominance¹¹⁾ and monopole dominance

12).

The abelian part of the field as defined by abelian projection, in the max abelian gauge, is a good approximation to full dynamics, and of it the contribution of monopoles is dominant. In a sense abelian dominance is expected, since, after maximal abelian projection the links are diagonal within 85%. However the fact that it happens is surely relevant.

Looking at symmetry, as we do, is a complementary approach.

Hopefully a picture will emerge from all these efforts, which will improve our understanding of the theory.

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